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LETTER TO THE EDITOR

Discrete symmetries and supersymmetry

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Abstract. The charge conjugation and parity transformation properties of the supersymmetry algebra with SU(N), of Dondi, are discussed. It is shown when and how this algebra may be modified to accommodate charge conjugation or parity invariance.

Recently, Dondi (1975) has generalized the supersymmetry algebra with SU(2) of Salam and Strathdee (1974) to a supersymmetry algebra with SU(N). He introduces supersymmetry generators S_{ai} and its Hermitian conjugate $\bar{S}_{\dot{a}}{}^{i}$. The first is an undotted SL(2, C) spinor and an SU(N) spinor, whilst the second is a dotted SL(2, C) spinor and an SU(N) conjugate spinor.

With $\mathscr{C}$ the charge conjugation operator, we define the charge conjugate supersymmetry generators by

$$\mathscr{C}S_{ai}\mathscr{C}^{-1} = S^{(c)}{}_{a}{}^{i} \tag{1}$$

and

$$\mathscr{C}\bar{S}_{\dot{a}}{}^{i}\mathscr{C}^{-1} = \bar{S}^{(c)}{}_{\dot{a}i}. \tag{2}$$

Note that, whilst charge conjugation does not alter the SL(2, C) representation, it transforms an SU(N) representation into its conjugate. Now it will be convenient to rewrite (1) and (2) by introducing the Dirac spinor and its conjugate†

$$S_{\alpha i} = \begin{pmatrix} S_{ai} \\ \bar{S}^{(c)\dot{a}}{}_{i} \end{pmatrix} \qquad \text{and} \qquad \bar{S}^{\alpha i} = (S^{(c)ai}\bar{S}_{\dot{a}}{}^{i}) \tag{3}$$

respectively. (1) and (2) then become

$$\mathscr{C}S_{\alpha i}\mathscr{C}^{-1} = C_{\alpha\beta}\bar{S}^{\beta i}. \tag{4}$$

Two distinct cases arise. In the case $N = 2$, since the spinor representation of SU(2) and its conjugate are equivalent, we may demand that

$$\bar{S}^{(c)\dot{a}}{}_{i} = \epsilon_{ij}\bar{S}^{\dot{a}j} \equiv \bar{S}^{\dot{a}}{}_{i}, \tag{5}$$

ϵ_{ij} being the lowering matrix for SU(2) spinor indices. It then follows from (1), (2) and (3) that

$$\mathscr{C}S_{\alpha i}\mathscr{C}^{-1} = (\mathrm{i}\gamma_5)_{\alpha}{}^{\beta}S_{\beta}{}^{i} \tag{6}$$

† Throughout this letter, the notation of Salam and Strathdee (1974), for γ matrices etc, is used.

and from (5) and (6) it follows that

$$S_{\alpha i} = \mathrm{i}\epsilon_{ij}(\gamma_5 C)_{\alpha\beta}\bar{S}^{\beta j}, \tag{7}$$

ie $S_{\alpha i}$ is a generalized Majorana spinor. In this way, the supersymmetry algebra with SU(2) of Salam and Strathdee (1974) is recovered from Dondi's sypersymmetry algebra. Thus, for $N = 2$, charge conjugation can be defined so that it transforms Dondi's supersymmetry algebra into itself.

However, for $N \geqslant 3$, the spinor representation of SU(N) and its conjugate are not equivalent. Thus, in this case, no demand analogous to (5) can be made, when it follows from (1) and (2) that charge conjugation does not transform Dondi's supersymmetry algebra into itself. The notion of a Majorana spinor cannot be generalized to $N \geqslant 3$.

Thus although Dondi's supersymmetry algebra is suitable for a discussion of theories with charge conjugation invariance when $N = 2$, it is not so suitable when $N \geqslant 3$.

A supersymmetry algebra with SU(N), which is transformed into itself under charge conjugation when $N \geqslant 2$, may be constructed with the supersymmetry generators (3). The commutation and anticommutation relations are as follows (Salam and Strathdee 1974):

$$\begin{aligned} \{S_{\alpha i}, S_{\beta j}\} &= 0 \\ [S_{\alpha i}, P_\mu] &= 0 \\ [S_{\alpha i}, J_{\mu\nu}] &= \tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta S_{\beta i} \\ [S_{\alpha i}, \boldsymbol{F}] &= \tfrac{1}{2}(\boldsymbol{\lambda})_i{}^j S_{\alpha j} \end{aligned} \tag{8}$$

and their conjugates, and

$$\{S_{\alpha i}, \bar{S}^{\beta j}\} = 2\delta_i{}^j(\gamma_\mu)_\alpha{}^\beta P^\mu,$$

these to be supplemented by the Poincaré and SU(N) algebras, generated respectively by P_μ, $J_{\gamma\rho}$ and $\boldsymbol{F}$. That this algebra is transformed into itself under charge conjugation follows immediately from (4).

A discussion, parallel to the above, of the parity transformation properties of Dondi's supersymmetry algebra with SU(N), may also be given. With $\mathscr{P}$ the parity operator, we may define

$$\mathscr{P}S_{\alpha i}\mathscr{P}^{-1} = (\gamma_0)_\alpha{}^\beta S_{\beta i} \tag{9}$$

with $S_{\alpha i}$ as in (3). Note that, whilst parity does not alter the SU(N) representation, it connects the dotted and undotted SL(2, C) representations. Thus parity also does not transform Dondi's supersymmetry algebra into itself when $N \geqslant 3$. However, when $N = 2$, (5) assures that parity, as defined by (9), transforms Dondi's supersymmetry algebra into itself. On the other hand, it follows immediately from (9) that parity transforms the algebra (8) into itself when $N \geqslant 2$.

When $N = 2$, the case of Salam and Strathdee, it is known that Dondi's supersymmetry algebra with SU(N) can accommodate charge conjugation or parity invariance. However, when $N \geqslant 3$, we have seen that this is not the case. It is of interest to note, though, as follows from (3), (4), (9) and the form of γ_0, that the operator $\mathscr{C}\mathscr{P}$ *does* transform Dondi's algebra into itself, even for $N \geqslant 3$.

A manifestation of the above difficulties with discrete symmetries is the impossibility of generalization of the notion of a Majorana spinor to $N \geqslant 3$. This noted, these difficulties have been circumvented in (8) by the construction of a supersymmetry algebra with SU(N), which is generated by the Dirac spinor $S_{\alpha i}$ and its conjugate $\bar{S}^{\alpha i}$.

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